## Framework

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<td>( \text{out}[b] = f_b(\text{in}[b]) )</td>
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<td>( \text{out}[b] = \emptyset )</td>
<td>( \text{in}[b] = \emptyset )</td>
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Other examples (e.g., Available expressions), defined in ALSU 9.2.6
Foundations of Data Flow Analysis

1. Meet operator
2. Transfer functions
3. Correctness, Precision, Convergence
4. Efficiency

• Reference: ALSU pp. 613-631
• Background: Hecht and Ullman, Kildall, Allen and Cocke[76]
A Unified Framework

- Data flow problems are defined by
  - Domain of values: $V$
  - Meet operator ($V \land V \varnothing V$), initial value
  - A set of transfer functions ($V \triangleright V$)

- Usefulness of unified framework
  - To answer questions such as correctness, precision, convergence, speed of convergence for a family of problems
    - If meet operators and transfer functions have properties $X$, then we know $Y$ about the above.

  - Reuse code
Meet Operator

• Properties of the meet operator
  • commutative: \( x \land y = y \land x \)
  • idempotent: \( x \land x = x \)
  • associative: \( x \land (y \land z) = (x \land y) \land z \)
  • there is a Top element \( T \) such that \( x \land T = x \)

• Meet operator defines a partial ordering on values
  • \( x \leq y \) if and only if \( x \land y = x \) \( (y \to x \text{ in diagram}) \)
    – Transitivity: if \( x \leq y \) and \( y \leq z \) then \( x \leq z \)
    – Antisymmetry: if \( x \leq y \) and \( y \leq x \) then \( x = y \)
    – Reflexitivity: \( x \leq x \)
Partial Order

- Example: let $V = \{x \mid \text{such that } x \subseteq \{d_1, d_2\}\}$, $\land = \cap$

- Top and Bottom elements
  - Top $T$ such that: $x \land T = x$
  - Bottom $\perp$ such that: $x \land \perp = \perp$

- Values and meet operator in a data flow problem define a semi-lattice:
  - there exists a $T$, but not necessarily a $\perp$.
- $x, y$ are ordered: $x \leq y$ then $x \land y = x$ (y -> x in diagram)
- what if $x$ and $y$ are not ordered?
  - $x \land y \leq x, x \land y \leq y$, and if $w \leq x, w \leq y$, then $w \leq x \land y$
One vs. All Variables/Definitions

• Lattice for each variable: e.g. intersection

\[
\begin{array}{c}
1 \\
\downarrow \\
0
\end{array}
\]

• Lattice for three variables:
Descending Chain

• Definition
  - The **height** of a lattice is the largest number of \( > \) relations that will fit in a descending chain.
    \[
    x_0 > x_1 > x_2 > \ldots
    \]

• Height of values in reaching definitions?
  - Height \( n \) – number of definitions

• Important property: **finite descending chain**

• Can an infinite lattice have a finite descending chain?
  - yes

• Example: Constant Propagation/Folding
  - To determine if a variable is a constant

• Data values
  - undef, ... -1, 0, 1, 2, ..., not-a-constant
Transfer Functions

• Basic Properties $f: V \rightarrow V$
  
  – Has an identity function
    
    • There exists an $f$ such that $f(x) = x$, for all $x$.

  – Closed under composition
    
    • if $f_1, f_2 \in F$, then $f_1 \cdot f_2 \in F$
Monotonicity

• A framework $(F, V, \land)$ is monotone if and only if
  • $x \leq y$ implies $f(x) \leq f(y)$
  
  • i.e. a “smaller or equal” input to the same function will always give a “smaller or equal” output

• Equivalently, a framework $(F, V, \land)$ is monotone if and only if
  • $f(x \land y) \leq f(x) \land f(y)$

  • i.e. merge input, then apply $f$ is small than or equal to apply the transfer function individually and then merge the result
Example

• Reaching definitions: \( f(x) = \text{Gen} \cup (x - \text{Kill}), \quad \land = \cup \)
  
  – Definition 1:
    • \( x_1 \leq x_2, \text{Gen} \cup (x_1 - \text{Kill}) \leq \text{Gen} \cup (x_2 - \text{Kill}) \)
  
  – Definition 2:
    • \( (\text{Gen} \cup (x_1 - \text{Kill})) \cup (\text{Gen} \cup (x_2 - \text{Kill})) \)
      = \( (\text{Gen} \cup ((x_1 \cup x_2) - \text{Kill})) \)

• Note: Monotone framework does not mean that \( f(x) \leq x \)
  
  • e.g., reaching definition for two definitions in program
  • suppose: \( f_x : \text{Gen}_x = \{d_1, d_2\} ; \text{Kill}_x = {} \)

• If input(second iteration) \leq input(first iteration)
  
  • result(second iteration) \leq result(first iteration)
Distributivity

- A framework \((F, V, \wedge)\) is **distributive** if and only if 
  - \(f(x \wedge y) = f(x) \wedge f(y)\)
  - i.e. merge input, then apply \(f\) is **equal to** apply the transfer function individually then merge result

- Example: Constant Propagation is NOT distributive
Data Flow Analysis

• Definition
  – Let $f_1, \ldots, f_m : \in F$, where $f_i$ is the transfer function for node $i$
    • $f_p = f_{n_k} \cdot \ldots \cdot f_{n_1}$, where $p$ is a path through nodes $n_1, \ldots, n_k$
    • $f_p = \text{identify function}$, if $p$ is an empty path

• Ideal data flow answer:
  – For each node $n$:
    $\bigwedge f_{p_i}(T)$, for all possibly executed paths $p_i$ reaching $n$.

\[
\text{if } \sqrt{y} \geq 0
\]

\[
\begin{array}{c}
x = 0 \\
x = 1
\end{array}
\]

• But determining all possibly executed paths is \textbf{undecidable}
Meet-Over-Paths (MOP)

- Error in the conservative direction
- **Meet-Over-Paths (MOP):**
  - For each node \( n \):
    
    \[
    \text{MOP}(n) = \bigwedge f_{p_i}(T), \text{ for all paths } p_i \text{ reaching } n
    \]
  
    \[-\text{a path exists as long there is an edge in the code}\]
  
    \[-\text{consider more paths than necessary}\]
  
    \[-\text{MOP} = \text{Perfect-Solution} \bigwedge \text{Solution-to-Unexecuted-Paths}\]
  
    \[-\text{MOP} \leq \text{Perfect-Solution}\]
  
    \[-\text{Potentially more constrained, solution is small}\]
    
    \[-\text{hence conservative}\]
  
    \[-\text{It is not safe to be } > \text{ Perfect-Solution!}\]

- **Desirable solution:** as close to MOP as possible
MOP Example

```
B1 if x == 1

B2 B3

B4 if x == 0

B5 B6

B7

Assume: B2 & B3 do not update x
```

Ideal: Considers only 2 paths
B1-B2-B4-B6-B7 (i.e., x=1)
B1-B3-B4-B5-B7 (i.e., x=0)

MOP: Also considers unexecuted paths
B1-B2-B4-B5-B7
B1-B3-B4-B6-B7
Solving Data Flow Equations

Example: Reaching definitions
- out[entry] = {}
- Values = {subsets of definitions}
- Meet operator: ∪
  - in[b] = ∪ out[p], for all predecessors p of b
- Transfer functions: out[b] = gen_b ∪ (in[b] - kill_b)

Any solution satisfying equations = Fixed Point Solution (FP)

Iterative algorithm
- initializes out[b] to {}
- if converges, then it computes Maximum Fixed Point (MFP):
  - MFP is the largest of all solutions to equations

Properties:
- FP ≤ MFP ≤ MOP ≤ Perfect-solution
- FP, MFP are safe
- in(b) ≤ MOP(b)
Partial Correctness of Algorithm

• If data flow framework is monotone, then if the algorithm converges, $\text{IN}[b] \leq \text{MOP}[b]$

• Proof: Induction on path lengths
  
  – Define $\text{IN}[\text{entry}] = \text{OUT}[\text{entry}]$
    and transfer function of entry = Identity function
  
  – Base case: path of length 0
    • Proper initialization of $\text{IN}[\text{entry}]$
  
  – If true for path of length $k$, $p_k = (n_1, ..., n_k)$, then true for path of length $k+1$: $p_{k+1} = (n_1, ..., n_{k+1})$
    • Assume: $\text{IN}[n_k] \leq f_{nk-1}(f_{nk-2}(...f_{n1}(\text{IN}[\text{entry}])))$
    • $\text{IN}[n_{k+1}] = \text{OUT}[n_k] \land ...$
      $\leq \text{OUT}[n_k]$
      $\leq f_{nk}(\text{IN}[n_k])$
      $\leq f_{nk-1}(f_{nk-2}(...f_{n1}(\text{IN}[\text{entry}])))$
Precision

• If data flow framework is **distributive**, then if the algorithm converges, \( \text{IN}[b] = \text{MOP}[b] \)

\[
\begin{align*}
\text{a} &= 2 \\
\text{b} &= 3 \\
\text{c} &= \text{a} + \text{b} \\
\text{a} &= 3 \\
\text{b} &= 2
\end{align*}
\]

• Monotone but not distributive: behaves as if there are additional paths
Additional Property to Guarantee Convergence

• Data flow framework (monotone) converges if there is a finite descending chain

• For each variable IN[b], OUT[b], consider the sequence of values set to each variable across iterations:

  – if sequence for in[b] is monotonically decreasing
    • sequence for out[b] is monotonically decreasing
      • (out[b] initialized to T)

  – if sequence for out[b] is monotonically decreasing
    • sequence of in[b] is monotonically decreasing
Speed of Convergence

• Speed of convergence depends on order of node visits

• Reverse “direction” for backward flow problems
Reverse Postorder

- **Step 1:** depth-first post order
  
  ```
  main() {
      count = 1;
      Visit(root);
  }

  Visit(n) {
      for each successor s that has not been visited
          Visit(s);
      PostOrder(n) = count;
      count = count + 1;
  }
  ```

- **Step 2:** reverse order
  
  For each node i
  
  ```
  rPostOrder = NumNodes - PostOrder(i)
  ```
Depth-First Iterative Algorithm (forward)

input: control flow graph CFG = (N, E, Entry, Exit)
/* Initialize */

out[entry] = init_value
For all nodes i
  out[i] = T
Change = True
/* iterate */

While Change {
  Change = False
  For each node i in rPostOrder {
    in[i] = \land(out[p]), for all predecessors p of i
    oldout = out[i]
    out[i] = f_i(in[i])
    if oldout \neq out[i]
      Change = True
  }
}
Speed of Convergence

• If cycles do not add information
  • information can flow in one pass down a series of nodes of increasing order number:
    • e.g., 1 -> 4 -> 5 -> 7 -> 2 -> 4 ...
  • passes determined by number of back edges in the path
    • essentially the nesting depth of the graph
  • Number of iterations = number of back edges in any acyclic path + 2
    • (2 are necessary even if there are no cycles)

• What is the depth?
  – corresponds to depth of intervals for “reducible” graphs
  – in real programs: average of 2.75
A Check List for Data Flow Problems

• **Semi-lattice**
  – set of values
  – meet operator
  – top, bottom
  – finite descending chain?

• **Transfer functions**
  – function of each basic block
  – monotone
  – distributive?

• **Algorithm**
  – initialization step (entry/exit, other nodes)
  – visit order: rPostOrder
  – depth of the graph
Conclusions

• Dataflow analysis examples
  – Reaching definitions
  – Live variables

• Dataflow formation definition
  – Meet operator
  – Transfer functions
  – Correctness, Precision, Convergence
  – Efficiency
CSC D70: Compiler Optimization
Loops

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University of Toronto
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The content of this lecture is adapted from the lectures of Todd Mowry and Phillip Gibbons
What is a Loop?

- **Goals:**
  - Define a loop in graph-theoretic terms (control flow graph)
  - Not sensitive to input syntax
  - A uniform treatment for all loops: DO, while, goto’s

- **Not every cycle is a “loop” from an optimization perspective**

- **Intuitive properties of a loop**
  - single entry point
  - edges must form at least a cycle
Formal Definitions

• Dominators
  – Node $d$ dominates node $n$ in a graph ($d \ dom \ n$) if every path from the start node to $n$ goes through $d$

  – Dominators can be organized as a tree
    • $a \rightarrow b$ in the dominator tree iff $a$ immediately dominates $b$
Dominance

x strictly dominates w (x sdom w) iff x dom w AND x ≠ w
Natural Loops

• Definitions
  – Single entry-point: header
    • a header dominates all nodes in the loop
  – A back edge is an arc whose head dominates its tail (tail -> head)
    • a back edge must be a part of at least one loop
  – The natural loop of a back edge is the smallest set of nodes that includes the head and tail of the back edge, and has no predecessors outside the set, except for the predecessors of the header.
Natural Loops - Example
Algorithm to Find Natural Loops

• Find the dominator relations in a flow graph

• Identify the back edges

• Find the natural loop associated with the back edge
1. Finding Dominators

- **Definition**
  - Node $d$ dominates node $n$ in a graph ($d \text{ dom } n$) if every path from the start node to $n$ goes through $d$

- **Formulated as MOP problem:**
  - Node $d$ lies on all possible paths reaching node $n$ $\Rightarrow d \text{ dom } n$
    - Direction:
    - Values:
    - Meet operator:
    - Top:
    - Bottom:
    - Boundary condition: start/entry node =
    - Initialization for internal nodes
    - Finite descending chain?
    - Transfer function:

- **Speed:**
  - With reverse postorder, most flow graphs (reducible flow graphs) converge in 1 pass
Example

\[
\text{OUT}[b]=\{b\} \cup (\cap_{p=\text{pre} \ (b)} \text{OUT}[p])
\]

\[
\begin{align*}
\text{OUT}[1] &= \{1\} \\
\text{OUT}[2] &= \{1,2\} \\
\text{OUT}[3] &= \{1,3\} \\
\text{OUT}[4] &= \{1,3,4\} \\
\text{OUT}[5] &= \{1,3,4,5\} \\
\text{OUT}[6] &= \{1,3,4,6\} \\
\text{OUT}[7] &= \{1,3,4,7\} \\
\text{OUT}[8] &= \{1,3,4,7,8\} \\
\text{OUT}[9] &= \{1,3,4,7,8,9\} \\
\text{OUT}[10] &= \{1,3,4,7,8,10\}
\end{align*}
\]

(No change in second iteration)
2. Finding Back Edges

- **Depth-first spanning tree**
  - Edges traversed in a depth-first search of the flow graph form a depth-first spanning tree

- **Categorizing edges in graph**
  - **Advancing (A)** edges: from ancestor to proper descendant
  - **Cross (C)** edges: from right to left
  - **Retreating (R)** edges: from descendant to ancestor (not necessarily proper)
Back Edges

• Definition
  – **Back edge**: t->h, h dominates t

• Relationships between graph edges and back edges

• Algorithm
  – Perform a depth first search
  – For each retreating edge t->h, check if h is in t’s dominator list

• Most programs (all structured code, and most GOTO programs) have **reducible** flow graphs
  – Retreating edges = back edges
Examples

All the retreating edges are back edges
3. Constructing Natural Loops

- The **natural loop of a back edge** is the smallest set of nodes that includes the head and tail of the back edge, and has no predecessors outside the set, except for the predecessors of the header.

- **Algorithm**
  - delete $h$ from the flow graph
  - find those nodes that can reach $t$
    (those nodes plus $h$ form the natural loop of $t \rightarrow h$)
Inner Loops

• If two loops do not have the same header:
  – they are either disjoint, or
  – one is entirely contained (nested within) the other
    • inner loop: one that contains no other loop.

• If two loops share the same header:
  – Hard to tell which is the inner loop
  – Combine as one
Preheader

• Optimizations often require code to be executed once before the loop
• Create a preheader basic block for every loop
Finding Loops: Summary

• Define loops in graph theoretic terms
• Definitions and algorithms for:
  – Dominators
  – Back edges
  – Natural loops
Backup Slides
Dominance Frontier

The Dominance Frontier of a node $x = \{ w \mid x \dom \pred(w) \AND \neg(x \sdom w)\}$

$x$ strictly dominates $w$ ($x \sdom w$) iff $x \dom w \AND x \neq w$
Dominance Frontier and Path Convergence